Commodity Trading Strategies in the Presence of Multiple Exchanges and Liquidity Constraints

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Abstract

The companies that consume commodities such as natural gas, crude oil and their products bear large risk exposure on the volatile price moves of these commodities. This is a main motivator of exploring this decision making problem for an inventory system which can both replenish and sell stock in multiple spot exchanges. Different from previous research in literature, this paper takes price-varying effect into account and characterizes the structure of optimal trade policies as market-dependent multi-level $(s; S; H)$ policies. It extends the work of [16] to the extent in which multiple suppliers are available simultaneously, and also [22] and [3] to the extent in which multiple discount opportunities coexist and vary over time. Thus, this work enriches the literature of generalized inventory ordering policies and lends itself to the energy industries, for instance, to optimize gas loading strategy of the gas fired generator.

摘要

某些商品，比如原油，天然气及其产品，由于它们的价格易变性，使消耗这些商品的商家需承受价格变化所带来的风险。因此，我们基于价格风险，来研究可在现货市场中买卖此类商品的库存系统的决策问题。与以往文献不同的是，本文的库存系统模型考虑了价格变动的影响，并发现解决该问题的最优策略是与市场相关的（s, S, H）策略。同时，基于多供应市场同时可用的假设，延伸了[16]的工作；本文也假设了多个折扣机会同时存在，并且这些折扣机会随时间变化，因此延伸了[22] 和[3]的工作。最后，本文扩充了传统库存系统的定购策略研究，这一有益补充也可应用于能源工业上。例如，本文结果可帮助优化天然气火力发电厂的天然气装填策略。

Acknowledgement

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Li Xu
2008 Hong Kong

To my family.

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Chapter 1

Introduction

Nowadays the companies that consume commodities such as natural gas, crude oil and their products bear large risk exposure due to the volatile prices moves of these commodities. It was reported in the news in [7] on June 7, 2008 that

"*After hitting its previous peak of \$133.17 a barrel last month, oil tumbled \$8.2 in nine days... (Yesterday) U.S. benchmark crude settled up by a record-setting \$10.75 to close at \$138.54 on the New York Mercantile Exchange.*"

However, one month later, the crude price declined sharply as reported in [17] on July 25,2008 that

"*Crude oil fell to a seven-week low as a report showed that OPEC is bolstering output to cut prices and fuel consumption in the U.S. and Asia drops. The OPEC increased output by* 200,000 *barrels a day in July,... oil slipped more than \$24 a barrel from the \$147.27 record on July 11, as fuel use declined.*"

The volatility of the commodities prices provides the companies that consume them incentive to hold or release inventory. In [1], this incentive is called *the speculative motive.* On one hand, there is a motive for holding inventory when the prices are rising rapidly. On the other hand, the companies may wish to release excess inventory when the prices are declining fast enough. Moreover, while the companies have speculative motive, they are not real speculators. That means they have liquidity constraints on borrowing, and they should not have short position in markets. Thus other than the risk from the uncertainty of the demand, it is also necessary to consider the risk of the speculative motive, which comes from the uncertainty of the price in the future. Hence, it is more complicated to control the inventory level in the environment where the commodities prices fluctuate.

There are many factors that could influence the price of the energy commodities. We take crude as an example. First, the objective factors such as the currency exchange rates, the output policies of the Organization of Petroleum Exporting Countries (OPEC) and other irritants including a sudden rise in political tempers in oil-rich Middle East will instigate the commodity price. Secondly, the current fluctuation of the commodity price could add fuel to the work of market speculators, which in return would aggravating the price volatility. Third, as indicated in [7], the surge of oil's price helped to send stocks amid fears that the U.S. economy could be in stagflation, i.e., the combination of inflation and slow growth. If that happened, it would add uncertainty to the oil's price in the future. Moreover, these above factors will work on for a long duration in the future due to the nonsubstitutability of the oil and the decreasing worldwide reserves.

The crude price impresses the prices of its products such as gasoline and diesel oil. With data from [2], Figure 1.1 displays the volatility of conventional gasoline price in New York Harbor spot exchange. Moreover, the setup cost incurred by trading in the exchange, which includes the cost to deliver the commodities, and the demand of the product of the companies are all influenced by the oil price. These also increase the risk faced by those companies and motivate us to explore the policy for the inventory systems, in order to determine how much commodities will be ordered or sold in the exchanges.

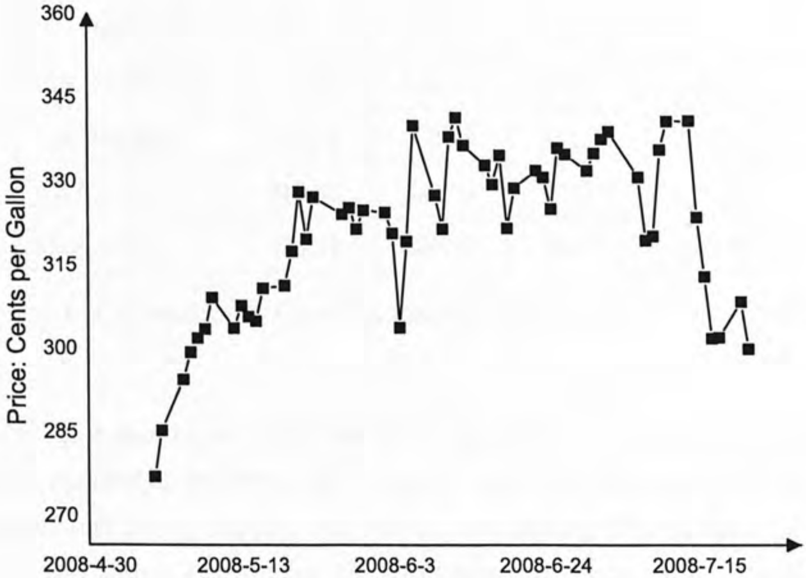

Figure 1.1: New York Harbor Conventional Gasoline Regular Spot Price, May-July 2008

To model the dynamic price, setup cost and demand, here we introduce a series of environment factors which affect the spot prices, setup cost and demand as in [16] and [19]. These factors are denoted by the *state-of-the-world*. They could be the fluctuating economic conditions, uncertain market conditions and so on, and they take effect by influencing the distribution of the random variables of the price, setup cost and demand. The *state-of-the-world* of a period, which is also referred as period state in short, is revealed at the beginning of the period, with future states unknown but following a Markov process.

Besides the volatility of the commodities prices, the existence of price gap between different spot exchanges is another motive of our work. As indicated in [5], companies are using different sources of supply with the ubiquitous presence of spot exchanges. Commonly there are three channels for commodities' transaction – contract markets, future mar-

Spot Exchange	06/25/08	06/26/08	06/27/08	06/30/08
NY Harbor	325.24	336.13	335.02	332.1
U.S. Gulf Coast	326.99	338.18	338.37	334.5
Los Angeles	347.74	353.43	351.12	348.25
ARA	316.56	330.02	330.02	330.02
Singapore	336.07	329.29	348.71	349.38

Table 1.1: Conventional Gasoline Regular Spot Prices (Cents/Gallon)

kets and spot markets. Different from the other two markets, the price in a spot market is determined by supply-demand balance. Rising price indicates that more supply is needed, and falling prices indicates that there is too much supply for the prevailing demand level. Since it is impossible for the spot exchanges to face the same supply-demand balance, the price gap arises. Table 1.1 illustrates the gap of conventional gasoline price, of which the data is from [2]. Therefore, our objective policy determines not only how much to trade in the spot exchange, and also which exchange(s) to trade at.

In summary, our interest in this work is a discrete-time, single-item, single-location inventory system which can trade in m_0 different spot exchanges. The system has the borrowing liquidity constraint that when it sells it could not sell what it doesn't hold. The period state is revealed at the beginning of the period, so are the prices of the exchanges. A trading decision is made then, and the trade does not influence the price in the exchange. After that the demand is realized, and the leftover is held and unmet amount is fully backlogged. We find that the optimal trading policy is the generalized (s, S, H) policy. An (s, S, H) policy means to make an order to bring the inventory level up to S when it is below s, and make a sale to bring the level down to S when it is above H. The generalized (s, S, H) policy is similar to the generalized (s, S) policy in [9] and [11], which has a series of threshold and target inventory levels. In our work these levels are dependent on the period state.

The remainder of the paper is organized as follows. In Chapter 2, we review the related literature, and then we introduce our model in Chapter3. In Chapter 4, we present the technical results and establish the optimal policy of single-period problem and finite-period problem. We further give an algorithm in this chapter to derive the optimal policy. Finally we conclude and suggest the direction for future research in Chapter 5.

□ **End of chapter.**

Chapter 2

Background Study

In [13] Scarf demonstrates the optimality of (s, S) policy for a class of finite-horizon periodic-review models with simplest type of non-linear ordering cost. Scarf's work gives one of the few fundamental results of inventory. And until today the dynamic (s, S) policy remains the best known result for dynamic control problems, although there have been some extensions in the past four decades.

Later Porteus in [9] analyzes inventory models with concave ordering cost, which could arise from multiple exchanges. He demonstrates that a generalized (s, S) policy, which is with multiple-threshold s and multiple-target S, is optimal for the periodic-review n-period problem with one-sided Pólya density. Then in [10] he finds the generalized (s, S) policy optimal for the problem with uniform demand densities.

There are some other literature focusing on multiple suppliers. [8] assumes the existence of multiple setup costs, and in [4], the authors solve the replenish problem where there is a supplier with high variable cost but negligible fixed cost and another supplier with lower variable cost but substantial fixed cost, and they prove that the optimal policy is either to purchase exclusively from one of the suppliers or to buy from both suppliers. Furthermore, some literature like [21] focus on the related problem of purchasing from both regular supplier and spot market. However, most of these literature assume the setup cost and variable cost are constant over time. Though in [21] the unit price at the spot market is dependent on the period n, that in the long-term supply contract is fixed, and they have not considered the setup cost on the spot market. Similar to our work, [15] assumes the entry cost to the spot market, but their problem is one-period. Moreover, in all of the above papers concerning spot markets they only consider the side of replenishment. Therefore, our model is closer to the real world.

At the side of dynamic economics environment, [19] and [16] consider an inventory model where the distribution of demand and the costs in successive period are dependent on a Markov Chain. [20] uses the Markovian stochastic process to decribe the evolution of the price state-random-variable. There are also some papers concerning the model with discount opportunities of ordering cost such as [22] and [3]. Our work is more general than them in that we consider more than one available supplier in our model and multiple discount opportunities.

To sum up, the model presented in the next chapter is more general than those by Porteus and [4] in that we consider state-dependent unit price and setup price, as well as demand distribution. And we extend the work of [16] to the extent in which multiple exchanges are available simultaneously, and also [22] and [3] to the extent in which multiple discount opportunities coexist and vary over time. Thus, this work enriches the literature of generalized inventory ordering policies and has wider application in the real world.

□ **End of chapter.**

Chapter 3

Model Formulation

In this chapter, We first characterize the trading cost function in our model, and then we introduce the basic notations and functions to solve the problem we bring forward, and finally we give the optimal equation which plays the key role in solving the problem.

We analyze a discrete-time, single-item, single-location inventory system which can trade a commodity in any exchanges from the set

$$M_0 = \{1, 2, \ldots, m_0\}.$$

The system has the borrowing liquidity constraint, thus when it sells it could only sell what is on hand. Different from the exist models with multiple suppliers such as [11] and [18], the unit price and setup cost fluctuate at each period in our model, and to some extent these costs could be treated as stochastic variables. Here we introduce a series of environment factors which affect the distribution of these random variables, and as in [16], these factors are represented by the *state-of-the-world*. To be more specific, if the state at period n is i, then the setup cost for exchange m is $\widetilde{K_{n,m}}(i)$, and the market unit price is $\widetilde{c_{n,m}}(i)$. We denote the realization of them at the beginning of period n as $K_{n,m}(i)$ and $c_{n,m}(i)$, they have supremums $\overline{K_{n,m}}(i)$ and $\overline{c_{n,m}}(i)$, and infimums $\underline{K_{n,m}}(i)$ and $\underline{c_{n,m}}(i)$. We assume that for different n, m

and i, $c_{n,m}(i)$ share the same supremum and infimum. Thus we could simplify $\overline{c_{n,m}}(i)$ and $\underline{c_{n,m}}(i)$ as $\overline{c}$ and $\underline{c}$.

3.1 Trading Cost Function

The trading cost comprises the setup cost and the variable cost, and the variable cost equals to the product of the unit price and trading quantity. The positive trading quantity means ordering decision, negative quantity means selling decision, and negative cost means profit. We assume if the system make an ordering decision then no selling decision can be made in the same period, and vice versa. This assumption is to eliminate the arbitrage opportunity and it makes sense. Though the trade may not influence the prices at the exchanges, the prices change so quickly that the decision under the altered prices can be considered as in the next period. Given the trading quantity, the decision that which exchange to trade at is based on their trading costs. In other words, an exchange will only be considered if its trading cost is no more than at any other exchanges. The trading cost function in period n is in the following form:

$$C_n(i,x) = \begin{cases} 0, & x = 0 \\ \min_m(K_{n,m}(i) + c_{n,m}(i)x), & x \neq 0 \end{cases} \tag{3.1}$$

where i is the period state and x is the trading quantity.

While the ordering cost function in [9] and [11] is similar to (3.1), they assume that the variable costs of M suppliers are descent order and setup cost on ascent order, i. e.,

$$c_1 > c_2 > \cdots > c_M \geq 0 \ , \ \textit{and} \ 0 \leq K_1 < K_2 < \cdots < K_M.$$

However, we don't use this assumption to retain a concave trading cost function for two reasons. First, under a competitive economic environment, the price and setup cost, especially of the commodity, fluctuate at a rapid speed. Therefore it is hard for the exchanges or

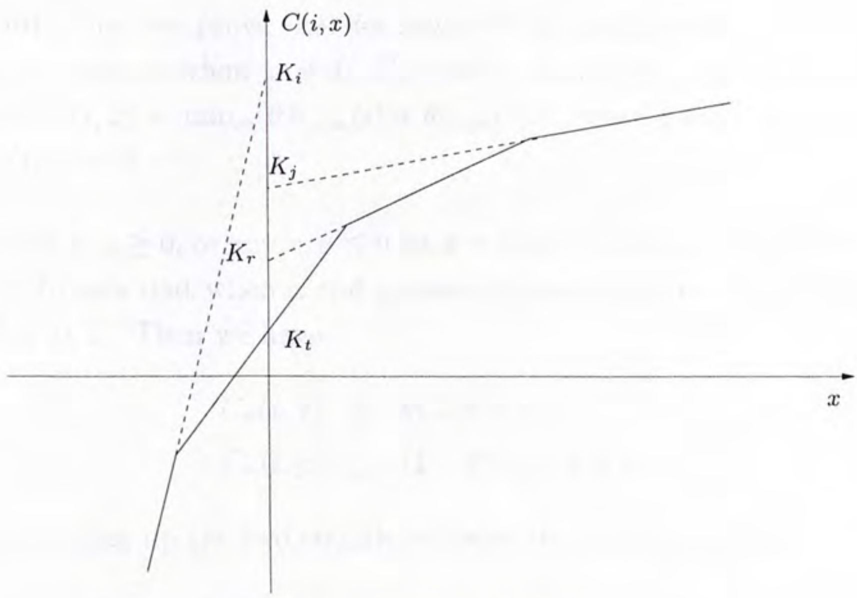

Figure 3.1: Trading Cost Function

suppliers to keep the cost well-ordered like this. Second, even without this assumption, we can also keep the trading cost function concave. In fact, under (3.1), there may exist some exchanges that would never be considered to trade at. For example, if the unit price and setup cost of Exchange 1, 2 and 3 have the relationships as following,

$$c_1 > c_2 > c_3 \quad and \quad K_2 > K_3 > K_1,$$

then the firm will consider Exchange 1 when selling and Exchange 1, 3 when purchasing, and will never consider Exchange 2. Thus (3.1) ensures the trading cost function concave in x.

We further give the Figure 3.1, which could show that the trading cost function is concave.

For this trading cost function, we have the following theorem.

Theorem 1 *The trading cost function $C_n(i,x)$ is subadditive, i. e. , $C_n(i,x+y) \leq C_n(i,x) + C_n(i,y)$ for any $x, y \geq 0$ and $x, y \leq 0$.*

Proof. First we prove that for any $\theta \in [0,1]$, $C_n(i,\theta x) \geq \theta C_n(i,x)$. That's because when $x \neq 0$, $C_n(i,\theta x) = \min_m(K_{n,m}(i) + \theta c_{n,m}(i)x)$ and $\theta C_n(i,x) = \min_m(\theta K_{n,m}(i) + \theta c_{n,m}(i)x)$; when $x = 0$, $C_n(i,\theta x) = \theta C_n(i,x) = 0$.

For any $x, y \geq 0$, or any $x, y \leq 0$ let $x = \theta(x+y)$ and $y = (1-\theta)(x+y)$. It is obvious that when x and y have the same sign, or one of them is 0, $\theta \in [0,1]$. Then we have

$$\begin{aligned} C_n(i,x) &\geq \theta C_n(i,x+y) \\ C_n(i,y) &\geq (1-\theta)C_n(i,x+y) \end{aligned}$$

By summing up the two equations above the theory applies. □

Theorem 1 says that if a company needs to replenish or sell some amount of inventory, it is better to do that in one time than in several times.

3.2 Notations and Optimality Equation

We introduce the notation as follows. N is the total number of periods. Since we consider the finite horizon problem, we have $N < \infty$. Denote $\{1, 2, \ldots, I\}$ as the finite collection of possible period states and i_n the state of period n. Thus, $\{i_n | n = 1, 2, \ldots, N\}$ composes a Markov chain, and we let $P = (p_{ij})_{I \times I}$ be its transition matrix. D_n is the demand in period n, $D_n \geq 0$ and let μ_n denote the mean of D_n. Here we suppose $E\{D_n | i_n = i\}$ has an upper boundary for any i. The demand is dependent on the period state, so we denote $\phi_{i,n}(\cdot)$ the conditional probability density function of D_n when $i_n = i$, and $\Phi_{i,n}(\cdot)$ the distribution function corresponding to $\phi_{i,n}(\cdot)$. Let x_n be the inventory level at the beginning of period before trading decision making, and y_n be the inventory level after trading decision is realized. Because the inventory system could not sell what it doesn't hold, $y_n \geq x_n$. $\mathcal{L}(x)$ is the holding and backlogging cost function incurred at the end of the period with ending inventory level x. Finally, we denote α to be the discount factor and $0 \leq \alpha \leq 1$.

The sequence is arranged as follows. The state of a period is revealed, so are the unit price and setup cost of each of the m_0 exchanges. Trading decision is made, order arrives or sale goes instantaneously, and the period's demand follows by. Unsatisfied demands are fully backlogged. It is important to point out that the period state is only revealed at the beginning of that period, with future states unknown but following a Markov process. Please refer to [12] for the characters of Markov process.

The objective is to minimize the total cost in this inventory system. Let $f_n(i, x)$ represent the minimum total cost from period n onward where the period begins with state i_n and inventory level x_n. The function $g_{n,m}$ is denoted as below:

$$g_{n,m}(i, x) := c_{n,m}(i)(x + \mu_n) + \mathcal{L}(x) + \alpha \sum_{j=1}^{I} p_{ij} f_{n+1}(j, x).$$

We define $G_{n,m} : \{1, \ldots, I\} \times R \to R$ by

$$\begin{aligned} G_{n,m}(i, y) &:= Eg_{n,m}(i, y - D_n) \\ &= \int_0^{\infty} g_{n,m}(i, y - x) d\Phi_{i,n}(x) \end{aligned}$$

It should be noticed that

$$G_{n,m}(i, y) - c_m(i)x$$

is the cost from period n onward while bringing inventory level from x to y and trading in period n is restricted to exchange m. By simple calculation, we get that

$$G_{n,m}(i, y) = c_{n,m}(i)y + L(i, y) + \alpha \sum_{j=1}^{I} p_{ij} E[f_{n+1}(j, y - D_n) | i_n = i], \quad (3.2)$$

where $L(i, x) = E[\mathcal{L}(x - D)|i]$.

Define the function $G^*_{n,m}$ by

$$G^*_{n,m}(i,x) = \min(G_{n,m}(i,x), \inf_{y\geq 0, y\neq x}[K_{n,m}(i) + G_{n,m}(i,y)]).$$

Thus, $G^*_{n,m}(i_n,x_n) - c_{n,m}(i)x_n$ is the minimum cost from period n onward while restricted to exchange m in period n. Moreover, it is worthy emphasizing the usual logic that if we are going to order/sell, we want to minimize $G_{n,m}(i,\cdot)$ moving to the right/left, and it should be checked whether the saving could cover the setup cost.

By now, we can get the optimality equation as

$$f_n(i_n,x_n) = \min_{m\in\{1,\ldots,m_0\}}[-c_{n,m}(i)x_n + G^*_{n,m}(i_n,x_n)]. \tag{3.3}$$

□ **End of chapter.**

Chapter 4

Optimal Policy

In this chapter we first establish some useful technical result and additional assumptions to derive the optimal policy. Then we focus on the single-period problem and investigate the possible structure of the optimal trading policy. Then we prove that the state-dependent generalized (s, S, H) policy is optimal for the finite-period problem under some assumptions, and we give an algorithm to derive the optimal policy at the end of this chapter.

4.1 Preliminary Assumption and Results

4.1.1 Generalized (s, S, H) Policy

We give the details of generalized (s, S, H) policy in the following definition.

Definition 1 *A decision rule δ is called generalized (s, S, H) policy if there exist m_1 and m_2 where*

$$S_1 \leq S_2 \leq \cdots \leq S_{m_1} \leq H_{m_1} \leq H_{m_1-1} \leq \cdots \leq H_2 \leq H_1$$
$$s_{m_2} \leq s_{m_2-1} \leq \cdots \leq s_2 \leq s_1 \leq S_{m_1+1} \leq \cdots \leq S_{m_1+m_2}$$

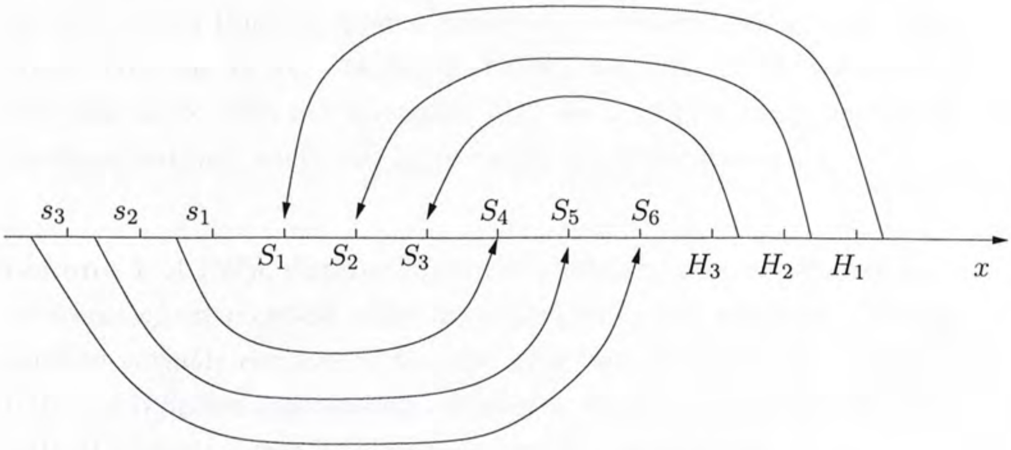

Figure 4.1: Illustration of A Generalized (s, S, H) Policy

and

$$s_{m_2} \le s_{m_2-1} \le \cdots \le s_1 \le H_{m_1} \le H_{m_1-1} \cdots \le H_1$$
$$S_1 \le S_2 \le \cdots \le S_{m_1} \le S_{m_1+1} \le \cdots \le S_{m_1+m_2}$$

such that

$$\delta(x) = \begin{cases} S_1 & \text{if } x \ge H_1 \\ S_i & \text{if } H_i \le x < H_{i-1} \text{ for } i = 2, \ldots, m_1 \\ S_{m_1+m_2} & \text{if } x \le s_{m_2} \\ S_{m_1+i} & \text{if } s_{i+1} \le x < s_i \text{ for } i = 1, 2, \ldots, s_{m_2-1} \\ x & \text{otherwise.} \end{cases}$$

Figure 4.1 displays the working of a generalized (s, S, H) policy for $m_1 = m_2 = 3$, where the arrows' bases indicate the inventory level before trading and their points indicate the level after trading. Here the trade includes buying and selling.

4.1.2 Pólya Distribution and Quasi-K-convex

Pólya distribution is PF_∞ distribution, and a Pólya random variable is a random variable with Pólya distribution. PF_n distributions, which are also called Pólya frequency functions of order n, have their elaborate definition in [6]. As the definitions are difficult to understand and have little with our objective, here we only give their equivalent characterizations, which are more intuitively understandable.

Lemma 1 *A Pólya distribution consists solely of translations and convolutions of exponentials, reflected exponentials and normals. A Pólya random variable consists of the sum of a translated normal, exponentials and reflected exponentials. A positive Pólya random variable consists of a positive translation of the sum of exponentials.*

This lemma is from [14], and to develop the positive Pólya random variables' smoothing properties we first have the following important lemma for the exponential random variables.

Lemma 2 *If f is continuous on the real axes, and g is defined by $g(x) := Ef(x - X)$, where X is an exponential random variable with mean $\frac{1}{\lambda}$, then g is continuously differentiable and*

$$g'(x) = \lambda[f(x) - g(x)].$$

This lemma implicates powerful smoothing property of exponential random variable. The derivative of g is proportional between f and itself. If g is below f, then g is increasing to approach f, and vice versa. The more distance between $f(x)$ and $g(x)$, the faster g get close to f. Thus g is smoother than f.

To develop the smoothing action of a positive Pólya random variable on the quasi-K-convex function, we first give the related definitions and properties.

Definition 2 *A function $f : R \to R$ is non-K-decreasing if there is*

$$f(x) \leq K + f(y)$$

for any $x \leq y$. If $f(-x)$ is non-K-decreasing then $f(x)$ is called non-K-increasing.

Definition 3 *A function $f : R \to R$ is quasi-K-convex with changeover at a if f is decreasing on $(-\infty, a]$ and non-K-decreasing on $[a, \infty)$.*

To establish our work we need the well-known properties of quasi-K-convex functions below, of which the proof could be found in the appendix of [9].

Proposition 1 *(a) If f is K-convex then f is quasi-K-convex.*
(b) If f is decreasing or non-K-decreasing then f is quasi-K-convex.
(c) If f is quasi-K-convex and $\gamma > 0$, then γf is quasi-γK-convex.
(d) The sum of a quasi-K-convex function with changeover at a and a quasi-k-convex function with changeover at a is quasi-$(K + k)$-convex with changeover at a.
(e) If $f(x)$ is quasi-K-convex with changeover at a, and $f(-x)$ is quasi-K-convex with changeover at $-b$, then $a \leq b$.

Lemma 3 *$f(x)$ is continuous and quasi-K-convex with changeover at a, and $f(-x)$ is quasi-K-convex with changeover at $-b$ for some $a \leq b$, if and only if it satisfies following properties:*
(1) $f(x)$ is decreasing on $(-\infty, a]$;
(2) $f(x)$ is increasing on $[b, +\infty)$;
(3) There exists $S \in [a, b]$ that minimize $f(x)$, and for any $x \in [a, b]$, $f(x) \leq f(S) + K$.

Proof. The "$\Longrightarrow$" is derived directly by the definition of quasi-K-convex. For the proof of "$\Longleftarrow$" part, suppose there exist $x, y \in [a, \infty)$, $x < y$, such that $f(x) > f(y) + K$, then y must be greater than b, since if $y \leq b$, then $a \leq x < y \leq b$, and we have $f(y) + k \geq f(S) + K \geq f(x)$. However, when $y > b$, if x is also greater than b, we have $f(y) + k > f(y) \geq f(x)$; if $x \leq b$, $f(y) + K \geq f(b) + K \geq f(S) + K \geq f(x)$.

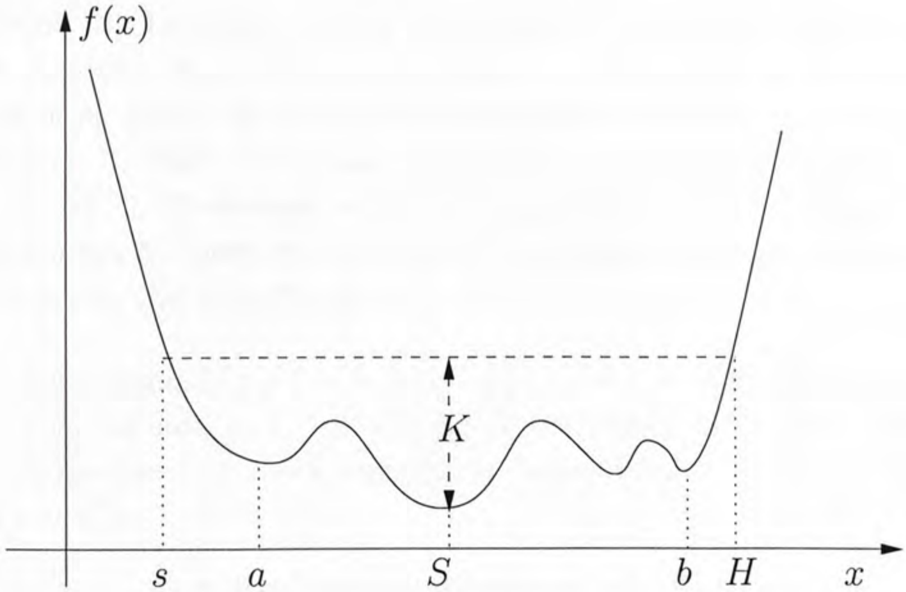

Figure 4.2: $f(x)$ Is Quasi-K-convex and $f(-x)$ Is Quasi-K-convex

Thus $f(x)$ is non-K-decreasing on $[a, +\infty)$. Similarly, $f(x)$ is non-K-increasing on $(-\infty, b]$. $f(x)$ is continuous and quasi-K-convex with changeover at a, and $f(-x)$ is quasi-K-convex with changeover at $-b$.

□

Figure 4.2 illustrates that $f(x)$ is continuous and quasi-K-convex with changeover at a, $f(-x)$ is quasi-K-convex with changeover at $-b$ and $a \leq b$. Please notice here that the changeover points are not unique. In this figure, any point in the interval $[s, a]$ is the changeover point of $f(x)$, and any point in $[-H, -b]$ is the changeover point of $f(-x)$.

Lemma 4 *If $f(x)$ is quasi-K-convex with changeover at a, and $f(-x)$ is quasi-K-convex with changeover at $-b$, D is an exponential random variable, then there exist some $a' \geq a$ and $b' \geq b$ such that $g(x) = Ef(x - D)$ is quasi-K-convex with changeover at a', $g(-x)$ is quasi-K-convex with changeover at $-b'$.*

Proof. By Lemma 3, $f(x)$ is decreasing on $(-\infty, a]$ and increasing on $[b, +\infty)$. There exist $S \in [a, b]$ that minimize $f(x)$, and for any $x \in [a, b]$, $f(S) + K \geq f(x)$. We divide the proof of this lemma into 3 parts. 1. There exists some $a' \geq a$ such that $g(x)$ is decreasing on $(-\infty, a']$. 2. There exists some $b' \geq b$ such that $g(x)$ is increasing on $[b', \infty)$ and 3. There exists $S' \in [a', b']$ that minimize $g(x)$, and for any $x \in [a', b']$, $g(x) \leq g(S') + K$.

1. By Lemma 2, $g'(x) = \lambda[f(x) - g(x)]$. Here D is an exponential random variable, so $\lambda > 0$. As $f(x)$ is decreasing on $(-\infty, a]$, there must be some $r \geq a$ such that $g(x)$ is decreasing on $(-\infty, r]$ too. We denote a' as

$$a' := max\{r | g(x) \text{ } is \text{ } decreasing \text{ } on \text{ } (-\infty, r]\}.$$

So $a' \geq a$ and $g(x)$ is decreasing on $(-\infty, a']$.

2. As $g(-x) = Ef(-x - D)$, we define $f(-x) = h(x)$, so $g(-x) = Eh(x + D) = Eh[x - (-D)]$, and $g'(-x) = -\lambda[h(x) - g(-x)] = -\lambda[f(-x) - g(-x)]$. Since $f(-x)$ is decreasing on$(-\infty, -b]$ and $-\lambda < 0$, there must be some $-r \leq -b$ such that $g(-x)$ is decreasing on $(-\infty, -r]$ too. We denote b' as

$$b' := min\{r | g(-x) \text{ } is \text{ } decreasing \text{ } on \text{ } (-\infty, -r]\}.$$

So $b' > b$ and $g(x)$ is increasing on $[b', \infty)$.

3. As $g(x)$ is continuous, decreasing on $(-\infty, a']$ and increasing on $[b', \infty)$, $g(x)$ must have a global minimizer on $[a', b']$, denote it as S'. So $g(S') = \lambda[f(S') - g(S')] = 0$. So $g(S') = f(S') \geq f(S)$.

Case 1. If there is no local maximizer of $g(x)$ on $[a', b']$, then $g(x)$ is decreasing on $(-\infty, S']$ and increasing on $[S', \infty)$. So $a' = S' = b'$, and it is clearly for any $x \in [a', b']$, $g(x) \leq g(S') + K$.

Case 2. If there is some local maximizers of $g(x)$ on $[a', b']$, then without loss of generality, choose the one that maximizes $g(x)$ on $[a', b']$ and denote it as S'_1. Then there must exist some local maximizers of $f(x)$

on $[a, b]$ and we choose the one that maximizes $f(x)$ on $[a, b]$ and denote it as S_1. Since $g'(S_1') = E[f(S_1') - g(S_1')] = 0$, $g(S_1') = f(S_1') \leq f(S) + K \leq g(S') + K$. Therefore, in this case for any $x \in [a', b']$, $g(x) \leq g(S') + K$.

To sum up, there exist some $a' \geq a$ and $b' \geq b$ such that $g(x) = Ef(x - D)$ is quasi-K-convex with changeover at a', $g(-x)$ is quasi-K-convex with changeover at b'. □

Corollary 1 *If $f(x)$ is quasi-K-convex with changeover at a, and $f(-x)$ is quasi-K-convex with changeover at $-b$, X is a positive Pólya random variable, then there exist some $a' > a$ and $b' > b$ such that $g(x) = Ef(x - X)$ is quasi-K-convex with changeover at a', $g(-x)$ is quasi-K-convex with changeover at $-b'$.*

Proof. By Lemma 1, X can be represented as a positive translation of the sum of exponentially distributed random variables. So its density can be represented recursively as a sequence of convolutions. Therefore, we can apply Lemma 4 recursively, and the corollary applies. □

Figure 4.3 illustrates a comparison of the figures of $f(x)$ in Figure 4.2 and $g(x) = Ef(x - X)$, and shows that Corollary 1 applies.

4.1.3 Assumptions

We have the following assumption of the demand.

Assumption 1 *At each period n the distribution of demand D_n is a positive Pólya random variable.*

Furthermore, we need the assumptions below to make the optimal policy turn out to be of generalized (s, S, H)-type.

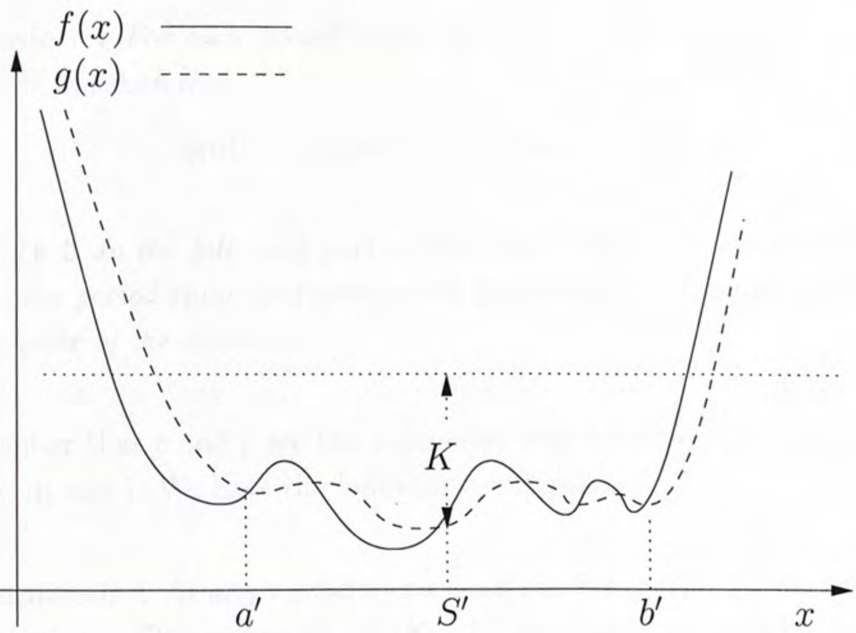

Figure 4.3: $f(x)$ and $g(x) = Ef(x - X)$

Assumption 2 *$L(i, x)$ is continuous in x for any period state $i \in \{1, 2, \dots, I\}$.*

Remark 1 *The continuity of $L(i, x) = E[\mathcal{L}(x - D_n)|i_n = i]$ does not need the holding and shortage cost function $\mathcal{L}(x)$ to be continuous on x. Specifically, $\mathcal{L}(x)$ could often be piecewise continuous on a countable set in the real world.*

Assumption 3 *$\mathcal{L}(x)$ gets sufficiently large for large absolute arguments of x, or equivalently,*

$$\lim_{x \to \infty} \mathcal{L}(x) \to +\infty. \tag{4.1}$$

Remark 2 *This assumption ensures it is never optimal to order or sell an infinite amount.*

We use the following function to rearrange the index order of the exchanges.

Definition 4 *For each period state $i \in \{1, 2, \ldots, I\}$, define function $i : M_0 \mapsto M_0$ such that*

$$c_{i(1)}(i) \geq c_{i(2)}(i) \geq \cdots \geq c_{i(m_0)}(i).$$

Remark 3 *In the following part of this thesis, when i is a number it means the period state, and when i is a function it is to rearrange the index order of the exchanges.*

Remember that $\overline{c}$ and $\underline{c}$ are the supremum and infimum of $c_{n,m}(i)$ for any n, m and i. We have the following assumptions.

Assumption 4 *At any period n, for each market m and period state i, $(1-\alpha)\underline{c}x + \mathcal{L}(x)$ is non-$(1-\alpha)\underline{K_{n,m}}(i)$-decreasing on $[a, +\infty)$, and $(\overline{c} - \alpha\underline{c})x + \mathcal{L}(x)$ is decreasing on $(-\infty, a]$. Here a is a scalar and $a \geq 0$.*

Assumption 5 *At any period n, for each market m and period state i, $(1-\alpha)\overline{c}x + \mathcal{L}(x)$ is non-$(1-\alpha)\underline{K_{n,m}}(i)$-increasing on $(-\infty, b]$, and $(\underline{c} - \alpha\overline{c})x + \mathcal{L}(x)$ is increasing on $[b, +\infty)$. Here b is a scalar and $b \geq a$.*

Remark 4 *The two assumptions above ensure that for any realization of the random variables $\widetilde{K_{n,m}}(i)$ and $\widetilde{c_{n,m}}(i)$, $K_{n,m}(i)$ and $c_{n,m}(i)$ fulfill that $(1-\alpha)c_{n,m}(i)x + \mathcal{L}(x)$ is non-$(1-\alpha)K_{n,m}(i)$-decreasing on $[a, +\infty)$, and non-$(1-\alpha)K_{n,m}(i)$-increasing on $(-\infty, b]$. Moreover, $(c_{n,i(1)}(i) - \alpha c_{n,i(m_0)}(i))x + \mathcal{L}(x)$ is decreasing on $(-\infty, a]$, and $(c_{n,i(m_0)}(i) - \alpha c_{n,i(1)}(i))x + \mathcal{L}(x)$ is increasing on $[b, +\infty)$.*

Finally, we have the assumption about the unit price and setup cost of the exchange markets.

Assumption 6 *For any period n and period state i, the realization of unit prices and setup cost of the exchange markets in that period are different from each other. In other words, for any $m \neq m'$, there exists $c_{n,m}(i) \neq c_{n,m'}(i)$ and $K_{n,m}(i) \neq K_{n,m'}(i)$.*

Remark 5 *This assumption does make sense. First, if at the beginning of a period two exchange markets have the same unit price and same setup cost, then these two markets could be considered as one; and if two exchanges with the same unit price have different setup cost, then the one with the higher setup cost would not be considered. Second, the setup cost mainly includes the market entry cost and transportation cost, which are always different for different exchange markets.*

Let $v_T(x)$ denote the terminal value if the leftover inventory at the end of the last period is x. We'll show that if v_T hold certain characteristics the generalized (s, S, H) policy would be optimal.

4.2 Single Period Problem

We consider the single period problem in this section and for convenience we rewrite the optimality equation (3.3) and other functions without the time index as follows

$$f(i,x) = \min_{m \in \{1,\ldots,m_0\}} [-c_m(i)x + G_m^*(i,x)] \tag{4.2}$$

where

$$G_m^*(i,x) = \min(G_m(i,x), \inf_{y \geq 0, y \neq x} [K_m(i) + G_m(i,y)], \tag{4.3}$$

$$\begin{aligned} G_m(i,x) &= \int_0^\infty g_m(i, y-x) d\Phi_{i,n}(x) \\ &= c_m(i)x + L(i,x) + \alpha E v_T(x-D), \end{aligned} \tag{4.4}$$

and

$$g_m(i,x) = c_m(i)(x+\mu) + \mathcal{L}(x) + \alpha v_T(x) \tag{4.5}$$

where D is the demand during the period and $L(i,x) = E[\mathcal{L}(x-D)|i]$.

The following lemma shows that if restricted to order from exchange market m it is optimal to follow a state dependent $(s_m(i), S_m(i), H_m(i))$ policy.

Lemma 5 *For any $m \in \{1, 2, \ldots, M\}$, if $G_m(i, x)$ is continuous and quasi-$K_m(i)$-convex with changeover at $a_m(i)$ on x for each $i \in \{1, 2, \ldots, I\}$, $a_m(i) \geq 0$, and $G_m(i, -x)$ is also quasi-$K_m(i)$-convex with changeover at $-b_m(i)$ on x for each i, then there exist $(s_m(i), S_m(i), H_m(i))$ and $s_m(i) \leq a_m(i) \leq S_m(i) \leq b_m(i) \leq H_m(i)$ for each i and m such that if the initial inventory level is below $s_m(i)$ it is optimal to order up to $S_m(i)$, if it is above $H_m(i)$ then sell down to $S_m(i)$ and do nothing otherwise.*

Proof. $G_m(i, \cdot)$ is quasi-$K_m(i)$-convex with changeover at $a_m(i)$, $G_m(i, -x)$ is also quasi-$K_m(i)$-convex with changeover at $-b_m(i)$. So we have $a_m(i) \leq b_m(i)$. By Lemma 3, $G_m(i, \cdot)$ is decreasing on $(-\infty, a_m(i)]$, increasing on $[b_m(i), \infty)$, non-$K_m(i)$-decreasing and non-$K_m(i)$-increasing on $[a_m(i), b_m(i)]$. We denote $S_m(i)$ the minimizer of $G_m(i, \cdot)$, so there must be $S_m(i) \in [a_m(i), b_m(i)]$ and $S_m(i) \geq 0$. Noting that there may more that one minimizer of of $G_m(i, \cdot)$, here we let

$$S_m(i) := \min\{x | x \text{ minimizes } G_m(i, \cdot) \textit{ and } x \geq a\}.$$

In other words, $S_m(i)$ is the least one of the minimizers. By Assumption 3, we have $\lim_{x \to \infty} G_m(i, x) = +\infty$, and by the third property of Lemma 3, there exist $s_m(i) \in (-\infty, a_m(i)]$ such that $G_m(i, s_m(i)) = G_m(i, S_m(i)) + K_m(i)$. And there exist $H_m(i) \in [b_m(i), \infty)$ such that $G_m(i, H_m(i)) = G_m(i, S_m(i)) + K_m(i)$. In summary $s_m(i) \leq a_m(i) \leq S_m(i) \leq b_m(i) \leq H_m(i)$.

Once we make a purchase decision, we want to minimize $G_m(i, \cdot)$ moving to the right, thus the best we can do is ordering up to $S_m(i)$. Similarly, once we make a sale decision, we want to minimize $G_m(i, \cdot)$ moving to the left, thus the best we can do is selling down to $S_m(i)$. However, if the saving is less than the setup cost $K_m(i)$ it is no worth to do this. Thus, if $x \leq s_m(i)$, $G_m(i, x) \geq G_m(i, s_m(i)) = G_m(i, S_m(i)) + K_m(i)$, it is optimal to order up to $S_m(i)$; if $s_m(i) \leq x \leq S_m(i))$, when $x \in (s_m(i), a)$, $G_m(i, x) \leq G_m(i, s_m(i)) = G_m(i, S_m(i)) + K_m(i)$, when $x \in [a, S_m(i)]$, $G_m(i)$ is non-$K_m(i)$-decreasing, also $G_m(i, x) \leq G_m(i, S_m(i)) + K_m(i)$ so it is better not to order; if $x \geq H_m(i)$, $G_m(i, x) \geq$

$G_m(i, H_m(i)) = G_m(i, S_m(i)) + K_m(i)$, it is optimal to sell down to $S_m(i)$; if $S_m(i) \leq x \leq H_m(i))$, when $x \in (b_m(i), H_m(i))$, $G_m(i, x) \leq G_m(i, H_m(i)) = G_m(i, S_m(i)) + K_m(i)$, when $x \in [S_m(i), b_m(i)]$, $G_m(i)$ is non-$K_m(i)$-increasing, also $G_m(i, x) \leq G_m(i, S_m(i)) + K_m(i)$ so it is better not to sell.

In summary, when the inventory level at the beginning of the period is below $s_m(i)$, it is optimal to order up to $S_m(i)$; if it is over $H_m(i)$, it is optimal to sell down to $S_m(i)$; otherwise it is better to do nothing. □

Proposition 2 $S_{i(1)}(i) \leq S_{i(2)}(i) \leq \cdots \leq S_{i(m_0)}(i)$ *for every* $i \in \{1, 2, \ldots, I\}$

Proof. It follows from (4.4) that for exchange $i(r)$ and $i(t)$,

$$G_{i(r)}(i, x) - G_{i(t)}(i, x) = [c_{i(r)} - c_{i(t)}]x.$$

Since $c_{i(r)} \geq c_{i(t)}$ for $r \leq t$, it follows that

$$\frac{\partial[G_{i(r)}(i, x) - G_{i(t)}(i, x)]}{\partial x} \geq 0,$$

which implicate the minimizer of $G_{i(r)}(i, x)$ is no more than that of $G_{i(t)}(i, x)$. Therefore, from the definition $i(\cdot)$ we have

$$S_{i(1)}(i) \leq S_{i(2)}(i) \leq \cdots \leq S_{i(m_0)}(i).$$

□

In [9] and [4], they assume that the exchange offering cheaper unit price always charged more on setup cost. It is reasonable to make this assumption in their work. That's because when only considering replenishment, a market which offers both higher setup cost and unit price could be intuitionally excluded. However, in our model we also consider to sell inventory to the market, so the market with higher setup cost and unit price may be considered. Therefore, we don't assume this kind of constraints between the unit price and setup cost of the exchange markets, only we need the Assumption 6 that different markets have different setup costs and unit prices. Yet we still apply the following lemma to pre-shrink the scope of right exchanges when the set of available exchanges is $M_0 = \{1, 2, \ldots, m_0\}$.

Lemma 6 *For any state $i \in \{1, 2, \ldots, I\}$, when the unit prices of the exchange markets are realized, there exist subsets M_1 and M_2 of M_0, such that when the company makes the sale decision, it only sell to the markets in M_1; when it makes purchase decision, it only purchase from the markets in M_2, and there are only one element in $M_1 \bigcap M_2$.*

Proof. We prove this lemma by constructing M_1 and M_2 to fulfill the the lemma. First we construct M_1. Put the market with the highest unit price of M_0 into M_1. Then do the following recursively. Compare the setup cost of the left market whose unit price is the highest in M_0 with that of the market whose unit price is the lowest in M_1. If the former is less than the latter, put the market with the former setup cost into M_1; otherwise drop it off M_0. In this way we get M_1 such that for any two markets in M_1, the one with higher unit price has a higher setup cost, and vice versa; and for any market left in M_0, there must be one in M_1 with a higher unit price but lower setup cost than that. Therefore, when the company makes the sale decision, it will not consider the left markets in M_0 but only sell to the markets in M_1. And it is easy to prove the one with the lowest setup cost of M_0 is in M_1.

We construct M_2 similarly. Put the market with the lowest unit price of M_0 into M_2. Then do the following recursively. Compare the setup cost of the left market whose unit price is the lowest in M_0 with that of the market whose unit price is the lowest in M_1. If the former is less than the latter, put the market with the former setup cost into M_2; otherwise drop it off M_0. In this way we get M_2 such that for any two markets in M_2, the one with higher unit price has a lower setup cost, and vice versa; and for any market left in M_0, there must be one in

M_1 with a lower unit price and lower setup cost than that. Therefore, when the company makes the purchase decision, it will not consider the left markets in M_0 but only sell to the markets in M_2. And it is easy to prove the one with the lowest setup cost of M_0 is in M_2.

Now it comes to prove the one with the lowest setup cost of M_0 is the only one element in $M_1 \bigcap M_2$. Assume it is Exchange 1. Suppose there is another element in $M_1 \bigcap M_2$, assume it is Exchange 2. Since Exchange 1, 2 are in M_1 and $K_1 < K_2$, then $c_1 < c_2$. However, since they are also in M_2, we have $c_1 > c_2$. It is a contradiction. Therefore, the only one element in $M_1 \bigcap M_2$ is the one with the lowest setup cost in M_0. Except this element, the unit price of other elements in M_1 is high than that in M_2. $\square$

The lemma below gives the conditions that ensure the generalized (s, S, H) policy is optimal. And in its proof process we provide a method not only to calculate the threshold of the policy, but also to decide which exchange to trade at for certain initial inventory level.

Lemma 7 *If $G_m(i, x)$ is continuous and quasi-$K_m(i)$-convex with changeover at $a_m(i)$ ($a_m(i) \geq 0$) on x for each $i \in \{1, 2, \ldots, I\}$ and each $m \in \{1, 2, \ldots, M\}$, and $G_m(i, -x)$ is also quasi-$K_m(i)$-convex with changeover at $-b_m(i)$ on x for each i and m, then there exists an optimal decision rule in form of state-dependent generalized (s, S, H) policy for the one-period problem.*

Proof. First we consider the purchase side. Fix the period state i, by Lemma 5 and Lemma 6, it is either optimal to order nothing or to make an order to bring the inventory level to $S_m(i)$ ($S_m(i) \geq 0$) for some $m \in M_2$. In other words, if we going to use exchange $m \in M_2$ to order we should bring the level to $S_m(i)$. Fix $t \in \{1, 2, \ldots, M\}$ and consider arbitrary $r < t$ and $r \in \{1, 2, \ldots, M\}$. Remember from Definition 5, $c_{i(r)}(i) \geq c_{i(t)}(i)$. While the initial inventory is at x, the saving from using exchange $i(t)$ compared using $i(r)$ is

$$K_{i(r)}(i)-K_{i(t)}(i)+(c_{i(t)}(i)-c_{i(r)}(i))x+G_{i(r)}(i,S_{i(r)}(i))-G_{i(t)}(i,S_{i(t)}(i)).$$

Let

$$s_{i(r),i(t)}(i) := \frac{K_{i(r)}(i) - K_{i(t)}(i) + G_{i(r)}(i,S_{i(r)}(i)) - G_{i(t)}(i,S_{i(t)}(i))}{c_{i(r)}(i) - c_{i(t)}(i)}. \tag{4.6}$$

Thus, $s_{i(r),i(t)}(i)$ is the initial inventory level at which there is no difference between using exchange $i(r)$ and $i(t)$.

When $c_{i(r)}(i) > c_{i(t)}(i)$, it is better to use exchange $i(t)$ than to use $i(r)$ iff the initial level $x \leq s_{i(r),i(t)}(i)$. We let

$$s^*_{i(t)}(i) := \min(s_{i(t)}(i), \min_{1\leq r<t} s_{i(r),i(t)}(i)).$$

Thus, if $x \leq s^*_{i(t)}(i)$ it is better to use exchange $i(t)$ than any exchanges whose variable cost is larger than $c_{i(t)}(i)$; and if $x \geq s^*_{i(t)}(i)$ it is better to do something other than use exchange $i(t)$. Therefore, if $s^*_{i(r)}(i) \leq s^*_{i(t)}(i)$, exchange $i(r)$ will never be used and can be ignored when identifying the optimal policy for this period.

From above it is in the same way to eliminate exchanges who will never be used at any period states. After eliminating any exchanges that need not be used, by re-indexing we have active exchange set $M_2' \leq M_2$ such that

$$s^*_{i(M_2')}(i) < \cdots < s^*_{i(1)}(i) \leq S^*_{i(1)}(i) \leq \cdots \leq S^*_{i(M_2')}(i).$$

If $x \geq s^*_{i(1)}(i)$, it is optimal to do nothing; if $s^*_{i(t)}(i) \leq x \leq s^*_{i(t+1)}(i)$, it is optimal to order up to $S^*_{i(t)}(i)$ in exchange $i(t)$ after re-indexing of M_2'. And please notice that here $s^*_{i(1)}(i)$ and $S_{i(1)}(i)^*$ are corresponding to the market with the lowest setup cost in M_0.

Similarly, for the sale side we have active exchange set $M_1' \leq M_1$ such that

$$S^{**}_{i(1)}(i) < \cdots < S^{**}_{i(M_1')}(i) \leq H^{**}_{i(M_1')}(i) \leq \cdots \leq S^{**}_{i(1)}(i).$$

If $x \leq H^{**}_{i(M_1')}(i)$, it is optimal to do nothing; if $H^{**}_{i(t+1)}(i) \leq x \leq H^{**}_{i(t)}(i)$, it is optimal to sell down to $S^{**}_{i(t)}(i)$ in exchange $i(t)$ after re-indexing of

M_1'. And please notice that here $H^{**}_{i(M_1')}(i)$ and $S^{**}_{i(M_1')}(i)$ are corresponding to the market with the lowest setup cost in M_0. By lemma 2, we have $s^*_{i(1)}(i) \le S^*_{i(1)}(i) = S^{**}_{i(M_1')}(i) \le H^{**}_{i(M_1')}(i)$. So the optimal policy is generalized (s, S, H)-type. Suppose there are m_1' elements in M_1' and m_2' elements in M_2'. We re-index the threshold and target inventory levels, and get

$$S_1 \le S_2 \le \cdots \le S_{m_1'} \le H_{m_1'} \le H_{m_1'-1} \le \cdots \le H_2 \le H_1$$
$$s_{m_2'} \le s_{m_2'-1} \le \cdots \le s_2 \le s_1 \le S_{m_1'+1} \le \cdots \le S_{m_1'+m_2'}$$

and

$$s_{m_2'} \le s_{m_2'-1} \le \cdots \le s_1 \le H_{m_1'} \le H_{m_1'-1} \cdots \le H_1$$
$$S_1 \le S_2 \le \cdots \le S_{m_1'} = S_{m_1'+1} \le \cdots \le S_{m_1'+m_2'}$$

These threshold and target inventory levels are all dependent of period state i. □

Denote V^* as the set of continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that for any m and i, for the scalar a and b define in Assumption 4 and 5, $0 \le a \le b$, the following characters apply.
(1) $\underline{c}x + f(x)$ is decreasing on $(-\infty, a]$;
(2) $\bar{c}x + f(x)$ is increasing on $[b, \infty)$;
(3) $c_m(i)x + f(x)$ is both non-$K_m(i)$-decreasing and non-$K_m(i)$-increasing on (a, b).

Theorem 2 *If $v_T \in V^*$, then a state-dependent generalized (s, S, H) policy is optimal for the single period problem.*

Proof. For each period state i, by rewriting the equation of $g_m(i, x)$ we get

$$g_m(i, x) = c_m(i)\mu + [(1-\alpha)c_m(i)x + \mathcal{L}(x)] + \alpha[c_m(i)x + v_T(x)]$$

By Assumption 4 and 5, $(1-\alpha)c_m(i)x + \mathcal{L}(x)$ is non-$(1-\alpha)K_m(i)$-decreasing on $(a, +\infty)$ and non-$(1-\alpha)K_m(i)$-increasing on $(-\infty, b)$.

Therefore $g_m(i,x)$ is non-$K_m(i)$-decreasing and non-$K_m(i)$-increasing on (a,b) by Proposition 1(c). Similarly $g_m(i,x)$ can be decomposed into

$$\begin{aligned} g_m(i,x) &= c_m(i)\mu + [(\bar{c} - \alpha\underline{c})x + \mathcal{L}(x)] \\ &\quad + (c_m(i) - \bar{c})x + \alpha[\underline{c}x + v_T(x)]. \end{aligned}$$

Because $(\bar{c} - \alpha\underline{c})x + \mathcal{L}(x)$ is decreasing on $(-\infty, a]$ by Assumption 4, $\underline{c}x + v_T(x)$ is decreasing on $(-\infty, a]$ as $v_T \in V^*$, and $(c_m(i) - \bar{c})x$ is decreasing on $\mathbb{R}$. Each term on the righthand side of the equation above is decreasing on $(-\infty, a]$. Also,

$$\begin{aligned} g_m(i,x) &= c_m(i)\mu + [(\underline{c} - \alpha\bar{c})x + \mathcal{L}(x)] \\ &\quad + (c_m(i) - \underline{c})x + \alpha[\bar{c}x + v_T(x)]. \end{aligned}$$

Since $(\underline{c} - \alpha\bar{c})x + \mathcal{L}(x)$ and $\bar{c}x + v_T(x)$ is increasing on $[b, \infty)$, $(c_m(i) - \underline{c})x$ is increasing on $\mathbb{R}$. Each term on the righthand side of the equation above is increasing on $[b, \infty)$.

Hence $g_m(i,x)$ is quasi-$K_m(i)$-convex with changeover at a, and $g_m(i,-x)$ is quasi-$K_m(i)$-convex with changeover at $-b$. By (4.4), Lemma 4 and Assumption 1, $G_m(i,x)$ is quasi-$K_m(i)$-convex with changeover at $a_m(i)$ for some $a_m(i) \geq a$, and $G_m(i,-x)$ is quasi-$K_m(i)$-convex with changeover at $b_m(i)$ for some $b_m(i) \geq b$ for each m and each i. Thus, by Lemma 7, a state-dependent generalized (s, S, H) policy is optimal for this problem. □

4.3 Finite-Period Problem

We consider the model over a finite horizon, i.e., $N < +\infty$, and suppose all the assumptions in Section 4.1 are also applied in this section. And we further have some other assumptions in this section.

For $i \in \{1, 2, \ldots, I\}$ and $n \in \{1, 2, \ldots, N\}$, denote

$$\overline{K_n}(i) := \max_m[\overline{K_{n,m}}(i)] \text{ and } \underline{K_n}(i) := \min_m[\underline{K_{n,m}}(i)]$$

We make a slight change on V^* in Section 4.2. For $i \in \{1, 2, \ldots, I\}$ and $n \in \{1, 2, \ldots, N\}$, denote $V^*_{n,i}$ as the set of continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that there exist two sequences of number $\{a_n(i)\}$ and $\{b_n(i)\}$, $a_n(i) \le b_n(i)$ and f satisfies
(1) $\underline{c}x + f(x)$ is decreasing on $(-\infty, a_n(i)]$;
(2) $\overline{c}x + f(x)$ is increasing on $[b_n(i), +\infty)$;
(3) Both $\underline{c}x + f(x)$ and $\overline{c}x + f(x)$ are non-$\underline{K_n}(i)$-decreasing and non-$\underline{K_n}(i)$-increasing on $(a_n(i), b_n(i))$.
And we denote $V^*_{N+1,i} := V^*$ and $f_{N+1}(i, x) = v_T(x)$ for any i. Thus $a_{N+1}(i) = a$, $b_{N+1}(i) = b$ and $\underline{K_{N+1}}(i) = \underline{K_N}(i)$ for any i.

Assumption 7 *For any period state $i, j \in \{1, 2, \ldots, I\}$ and any $n \in \{1, 2, \ldots, N-1\}$, we have*

$$\underline{K_n}(i) \ge \overline{K_{n+1}}(j).$$

Assumption 7 ensures that the setup cost is decreasing as the time lapses. With $\overline{a_n} := \max_i[a_n(i)]$, $\underline{a_n} := \min_i[a_n(i)]$, $\overline{b_n} := \max_i[b_n(i)]$ and $\underline{b_n} := \min_i[b_n(i)]$, we further need to make slight changes on Assumption 4 and 5.

Assumption 8 *At any period n, $n \in \{1, 2, \ldots, N+1\}$, for each market m and period state i, $(1-\alpha)\underline{c}x + \mathcal{L}(x)$ is non-$(1-\alpha)\underline{K_n}(i)$-decreasing on $[\underline{a_n}, +\infty)$, and $(\overline{c} - \alpha\underline{c})x + \mathcal{L}(x)$ is decreasing on $(-\infty, \underline{a_n}]$.*

Assumption 9 *At any period n, $n \in \{1, 2, \ldots, N+1\}$, for each market m and period state i, $(1-\alpha)\overline{c}x + \mathcal{L}(x)$ is non-$(1-\alpha)\underline{K_n}(i)$-increasing on $(-\infty, \overline{b_n}]$, and $(\underline{c} - \alpha\overline{c})x + \mathcal{L}(x)$ is increasing on $[\overline{b_n}, +\infty)$.*

Remark 6 *Assumption 8 and 9 are stricter than Assumption 4 and 5, though they appear to be vague. It is because that we have not given the explicit expression of a_n and b_n by now. However, if a_n and b_n are not infinite and have limitary for any n, the two assumptions above are* *still reasonable. In the lemma below we give the explicit expression of a_n and b_n and indicate they are not infinite.*

Lemma 8 *If for some $n \in \{1, 2, \dots, N\}$, $f_{n+1}(i, \cdot) \in V^*_{n+1,i}$ for any $i \in \{1, 2, \dots, I\}$, then the following properties hold.*
(a) $g_{n,m}(i, x)$ is quasi-$K_{n,m}(i)$-convex with changeover at $a'_{n,m}(i)$, and $g_{n,m}(i, -x)$ is quasi-$K_{n,m}(i)$-convex with changeover at $-b'_{n,m}(i)$ for each i and m.
(b) $G_{n,m}(i, x)$ is quasi-$K_m(i)$-convex with changeover at $a_{n,m}(i)$ and $G_{n,m}(i, -x)$ is quasi-$K_m(i)$-convex with changeover at $-b_{n,m}(i)$ for each i and m. Here $a_{n,m}(i) \geq a'_{n,m}(i)$ and $b_{n,m}(i) \geq b'_{n,m}(i)$.
(c) There exist a state-dependent generalized (s, S, H) policy that is optimal in the nth period.
(d) $f_n(i, \cdot) \in V^*_{n,i}$ for any $i \in \{1, 2, \dots, I\}$.

Proof. (a) Rewrite $g_{n,m}(i, x)$ as

$$g_{n,m}(i,x) = c_{n,m}(i)\mu + [(\underline{c} - \alpha\overline{c})x + \mathcal{L}(x)] + \alpha \sum_{j=1}^{I} p_{ij}[\underline{c}x + f_{n+1}(j,x)] + [c_{n,m}(i) - \overline{c}]x.$$

Then it is easy to prove that $g_{n,m}(i, x)$ is decreasing on $(-\infty, \underline{a_{n+1}}]$. Similarly $g_{n,m}(i, x)$ is increasing on $[\overline{b_{n+1}}, +\infty)$. $g_{n,m}(i, x$ could also be written as

$$g_{n,m}(i,x) = c_{n,m}(i)\mu + [(1-\alpha)c_{n,m}(i)x + \mathcal{L}(x)] + \alpha \sum_{j=1}^{I} p_{ij}[c_{n,m}(i)x + f_{n+1}(j,x)].$$

As $c_{n,m}(i)x + f_{n+1}(j, x)$ is non-$\underline{K_{n+1}}(j)$-decreasing and non-$\underline{K_{n+1}}(j)$-increasing on $(a_{n+1}(j), b_{n+1}(j))$, it is also non-$\underline{K_n}(i)$-decreasing and non-$\underline{K_n}(i)$-increasing on $(\overline{a_{n+1}}, \underline{b_{n+1}})$. Moreover, by Assumption 8 and 9, $(1 - \alpha)c_{n,m}(i)x + \mathcal{L}(x)$ is non-$(1 - \alpha)\underline{K_n}(i)$-decreasing and non-$\underline{K_n}(i)$-increasing on $(\underline{a_{n+1}}, \overline{b_{n+1}})$. Therefore, $g_{n,m}$ is both non-$K_{n,m}(i)$-decreasing and non-$K_{n,m}(i)$-increasing on $(\overline{a_{n+1}}, \underline{b_{n+1}})$. Further more, it could be proved that $g_{n,m}(i, x$ is also quasi-$\underline{K_n}(i)$-convex on $[\underline{a_{n+1}}, \overline{a_{n+1}}]$ and $[\underline{b_{n+1}}, \overline{b_{n+1}}]$. Therefore, there exist $a'_{n,m}(i) \in [\underline{a_{n+1}}, \overline{a_{n+1}}]$ and $b'_{n,m}(i) \in [\underline{b_{n+1}}, \overline{b_{n+1}}]$, such that $g_{n,m}(i, x)$ is quasi-$K_{n,m}(i)$-convex with changeover at $a'_{n,m}(i)$, and $g_{n,m}(i,-x)$ is quasi-$K_{n,m}(i)$-convex with changeover at $-b'_{n,m}(i)$.

The proof of (b) and (c) follows directly from Corollary 1 and Lemma 7.

(d) By (b), a state-dependent generalized (s,S,H) policy is optimal. By Lemma 6 and without loss of generality, we suppose that the company sell the inventory to the markets $\{1,2,\ldots,m_1(i)\}$ and order from $\{m_1(i), m_1(i)+1,\ldots,m_1(i)+m_2(i)\}$. Here we denote $m_0(i)=m_1(i)+m_2(i)$, $c_{n,1}(i) > c_{n,2}(i) > \ldots > c_{n,m_1(i)}(i) > \ldots > c_{n,m_1(i)+m_2(i)}(i)$, and the state-dependent thresholds and targets of this policy are

$$s_{m_2(i)}(i) \le \cdots \le s_1(i) \le H_{m_1(i)}(i) \le \cdots \le H_1(i)$$
$$S_1(i) \le S_2(i) \le \cdots \le S_{m_1(i)}(i) \le S_{m_1(i)+1}(i) \le \cdots \le S_{m_1(i)+m_2(i)}(i).$$

Thus, we have

$$f_n(i,x) = \begin{cases} -c_{n,1}(i)x + K_{n,1}(i) + G_{n,1}(i,S_1(i)), & x > H_1(i) \\ -c_{n,2}(i)x + K_{n,2}(i) + G_{n,2}(i,S_2(i)), & H_2(i) < x \le H_1(i) \\ \cdots\cdots & \cdots\cdots \\ -c_{n,m_1(i)}(i)x + K_{n,m_1(i)}(i) + G_{n,m_1(i)}(i,S_{m_1(i)}(i)), & H_{m_1(i)}(i) < x \le H_{m_1(i)-1}(i \\ -c_{n,m_1(i)}(i)x + K_{n,m_1(i)}(i) + G_{n,m_1(i)}(i,S_{m_1(i)}(i)), & s_2(i) \le x < s_1(i) \\ \cdots\cdots & \cdots\cdots \\ -c_{n,m_0(i)}(i)x + K_{n,m_0(i)}(i) + G_{n,m_0}(i,S_{m_0(i)}(i)), & x < s_{m_2(i)}(i) \\ L(i,x) + \alpha\sum_{j=1}^{I} p_{ij}Ef_{n+1}(j,x-D_n), & otherwise \end{cases} \tag{4.7}$$

By Assumption 2 and the derivation of these thresholds in Lemma 7, $f_n(i,x)$ is continuous. Let $a_n(i)=s_1(i)$, $b_n(i)=H_{m_1(i)}(i)$, thus $b_n(i)\ge a_n(i)$. We divide the remaining work into three steps: 1. $\bar{c}x+f_n(i,x)$ is increasing on $[b_n(i),+\infty)$; 2. $\underline{c}x+f_n(i,x)$ is decreasing on $(-\infty,a_n(i)]$; 3. Both $\underline{c}x+f_n(i,x)$ and $\bar{c}x+f_n(i,x)$ are non-$\underline{K_n}(i)$-decreasing and non-$\underline{K_n}(i)$-increasing on $(a_n(i),b_n(i))$.

1. Since $b_n(i)=H_{m_1(i)}(i)$, $f_n(i,x)$ is piecewise linear and decreasing

when $x \geq b_n(i)$, and none of the slopes are less than $-\bar{c}$. Therefore, $\bar{c}x + f_n(i,x)$ is increasing on the interval $[b_n(i), +\infty)$.

2. Since $a_n(i) = s_1(i)$, $f_n(i,x)$ is piecewise linear and decreasing when $x \leq a_n(i)$, and none of the slopes are greater than $-\underline{c}$. Therefore, $\underline{c}x + f_n(i,x)$ is decreasing on the interval $(-\infty, a_n(i)]$.

3. We only need to prove that $c_{n,m}(i)x + f_n(i,x)$ is non-$K_{n,m}(i)$-decreasing and non-$K_{n,m}(i)$-increasing on $(a_n(i), b_n(i))$ for any m.

Let $\omega_n(i,x) := L(i,x) + \alpha \sum_{j=1}^{I} p_{ij} E[f_{n+1}(j, x - D_n)|i_n = i]$, rewrite $f_n(i,x)$ as

$$f_n(i,x) = \inf_{z \geq 0}[C_n(i, z - x) + \omega_n(i,z)].$$

$C_n(i,z)$ is trading cost function that

$$C_n(i,z) = \begin{cases} 0, & z = 0 \\ \min_m (K_{n,m(}i) + c_{n,m}(i)z), & z \neq 0 \end{cases}$$

When $x \in (a_n(i), b_n(i))$, $s_1(i) < x < H_{m_1(i)}(i)$, thus the optimal decision is to do nothing. Then for any $x, y \in (a_n(i), b_n(i))$ and $y \geq x$,

$$\begin{aligned} f_n(i,x) &= \inf_{z \geq 0}[C_n(i, z - x) + \omega_n(i,z)] \\ &= \inf_{z \geq x}[C_n(i, z - x) + \omega_n(i,z)] \\ &\leq \inf_{z \geq y}[C_n(i, z - x) + \omega_n(i,z)] \\ &\leq \inf_{z \geq y}[C_n(i, z - y) + C_n(i, y - x) + \omega_n(i,z)] \\ &= \inf_{z \geq y}[C_n(i, z - y) + \omega_n(i,z)] + C_n(i, y - x) \\ &\leq f_n(i,y) + K_{n,m}(i) + c_{n,m}(i)(y - x) \end{aligned}$$

Here the first inequality exists because the constraint on $z \geq y$ limits the number of choice. The second inequality exists due to the characters of subaddtivite function by Theorem 1, and the last one due to the definition of trading cost function. Thus, for any m and any realization $c_{n,m}(i)$ and $K_{n,m}(i)$, $c_{n,m}(i)x + f_n(i,x)$ is non-$K_{n,m}(i)$-decreasing on $(a_n(i), b_n(i))$. Similarly, for any $x, y \in (a_n(i), b_n(i))$ and $y \le x$ we have

$$\begin{aligned}
f_n(i,x) &= \inf_{z\ge 0}[C_n(i,z-x)+\omega_n(i,z)] \\
&\le \inf_{0\le z\le y}[C_n(i,z-x)+\omega_n(i,z)] \\
&\le \inf_{0\le z\le y}[C_n(i,z-y)+C_n(i,y-x)+\omega_n(i,z)] \\
&= \inf_{0\le z\le y}[C_n(i,z-y)+\omega_n(i,z)]+C_n(i,y-x) \\
&\le f_n(i,y)+K_{n,m}(i)+c_{n,m}(i)(y-x)
\end{aligned}$$

Therefore, for any m, $c_{n,m}(i)x + f_n(i,x)$ is non-$K_{n,m}(i)$-increasing on $(a_n(i), b_n(i))$ for any realization $K_{n,m}(i)$, and $c_{n,m}(i)x + f_n(i,x)$ is non-$\underline{K_n}(i)$-increasing on $(a_n(i), b_n(i))$.

By now we have proved that $f_n(i,x)$ satisfies that 1. $\bar{c}x + f_n(i,x)$ is increasing on $[b_n(i), +\infty)$; 2. $\underline{c}x + f_n(i,x)$ is decreasing on $(-\infty, a_n(i)]$; 3. Both $\underline{c}x + f_n(i,x)$ and $\bar{c}x + f_n(i,x)$ are non-$\underline{K_n}(i)$-decreasing and non-$\underline{K_n}(i)$-increasing on $(a_n(i), b_n(i))$. Thus $f_n(i,x) \in V^*_{n,i}$. □

As indicated in the proof of Lemma 8, $a_n(i)$ is the maximal ordering threshold and $b_n(i)$ is the minimal selling threshold. Thus $a_n(i)$ and $b_n(i)$ are dependent with the realization of unit price and setup cost, i.e., $c_{n,m}(i)$ and $K_{n,m}(i)$. As $c_{n,m}(i)$ and $K_{n,m}(i)$ have boundaries, $a_n(i)$ and $b_n(i)$ are not infinite. Yet Assumption 8 and 9 have high restriction on $\mathcal{L}(x)$ and the boundaries of $c_{n,m}(i)$ and $K_{n,m}(i)$.

Theorem 3 *If $v_T(\cdot) \in V^*_{N+1}$, Assumption 1-3 and 6-9 hold, then a state-dependent generalized (s, S, H) policy is optimal in any periods.*

Proof. It applies by induction. □

By now we have characterized the optimal policy of this problem to find out which exchange market to trade at and how much to trade, and we illustrate the Optimality Cost Equation in Figure 4.4.

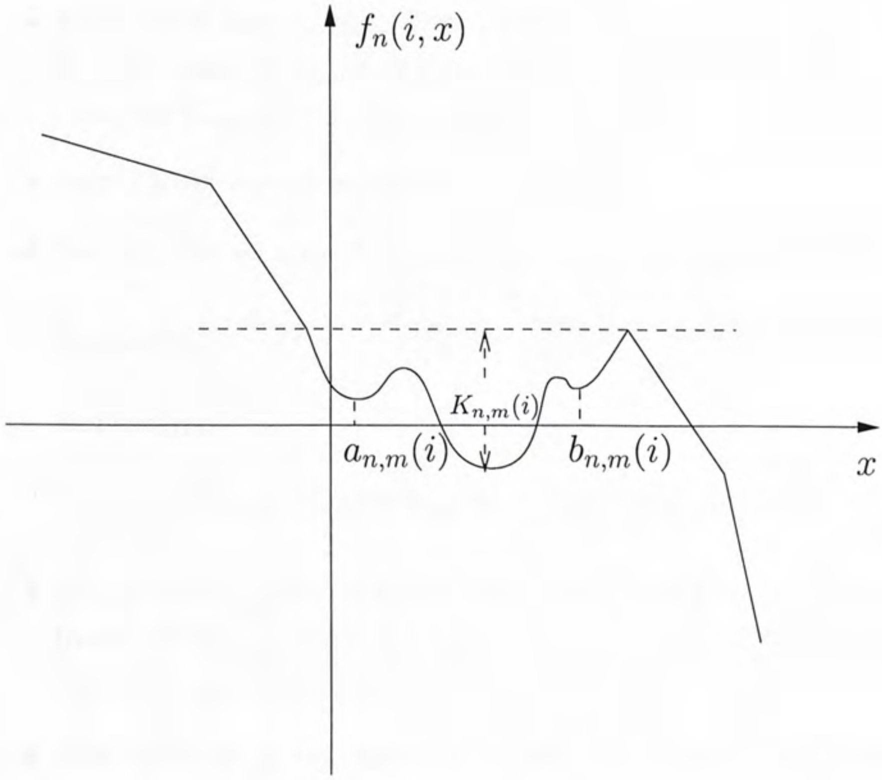

Figure 4.4: Optimality Equation

4.4 The Algorithm

In this section, we give an algorithm to compute the thresholds and targets of the optimal (s, S, H) policy and the optimal cost.

For any fixed i, $i \in \{1, 2, \ldots, I\}$, we first get M_1 and M_2 in Lemma 6 by the method referred in its proof, and re-index the exchanges such that $c_1(i) > c_2(i) > \cdots > c_{m_1}(i)$ is in M_1, $c_{m_1+1}(i) > c_{m_1+2}(i) > \cdots > c_{m_1+m_2}(i)$ is in M_2.

- *Step 0* (Initialization) Let set $U(n, i) = \emptyset$, $B(n, i) = \emptyset$. $M_1'(n, i) = \emptyset$ and $M_2'(n, i) = \emptyset$. Let $f_{N+1}(i, x) = v_T(x)$ for any $i \in \{1, 2, \ldots, I\}$. Set $i = 1$ and $n = N$.

- *Step 1* For any m that $1 \leq m \leq m_1$, compute $G_{n,i(m)}(i,x)$, $S_{n,i(m)}(i)$ and $H_{n,i(m)}(i)$; for any m that $m_1+1 \leq m \leq m_1+m_2$, compute $G_{n,i(m)}(i,x)$, $S_{n,i(m)}(i)$ and $s_{n,i(m)}(i)$

- *Step 2.0* Set $m=1$ and $R_1=\{+\infty\}$.

- *Step 2.1* For $m < m' \leq m_1$ and any $i \in \{1,2,\ldots,I\}$, compute

$$H_{n,m,m'}(i) := \frac{K_{n,m}(i) - K_{n,m'}(i) + G_{n,m}(i, S_{n,m}(i)) - G_{n,m'}(i, S_{n,m'}(i))}{c_{n,m}(i) - c_{n,m'}(i)}$$

  and compute

$$H^*_{n,m}(i) = \max[H_{n,m}(i), \min_{m<m'\leq m_1}(H_{n,m,m'}(i))].$$

- *Step 2.2* If $H^*_{n,m}(i) \geq \min R_1$, set $m=m+1$; if $H^*_{n,m}(i) < \min R_1$, then put $H^*_{n,m}(i)$ into R_1, $(S_{n,m}(i), H^*_{n,m}(i))$ into $U(n,i)$ and m into $M'_1(n,i)$, and set $m=m+1$.

- *Step 2.3* If $m \leq m_1$, go back to Step 2.1; if $m > m_1$, compute $f_n(i,x)$ with the threshold of ordering policy in $U(n,i)$ and corresponding exchanges in $M'_1(n,i)$ and go to Step 3.0.

- *Step 3.0* Set $m = m_1+m_2$ and $R_2=\{-\infty\}$.

- *Step 3.1* For $m_1+1 \leq m' < m$ and any $i \in \{1,2,\ldots,I\}$, compute

$$s_{n,m,m'}(i) := \frac{K_{n,m}(i) - K_{n,m'}(i) + G_{n,m}(i, S_{n,m}(i)) - G_{n,m'}(i, S_{n,m'}(i))}{c_{n,m}(i) - c_{n,m'}(i)}$$

  and compute

$$s^*_{n,m}(i) = \min[s_{n,m}(i), \min_{m_1+1\leq m'<m}(s_{n,m,m'}(i))].$$

- *Step 3.2* If $s^*_{n,m}(i) \leq \min R_2$, set $m=m+1$; if $H^*_{n,m}(i) > \min R_2$, then put $s^*_{n,m}(i)$ into R_2, $(s^*_{n,m}(i), S_{n,m}(i))$ into $B(n,i)$ and m into $M'_2(n,i)$, and set $m=m-1$.

- *Step 3.3* If $m \geq m_1+1$, go back to Step 2.1; if $m < m_1+1$, compute $f_n(i,x)$ with the threshold of ordering policy in $B(n,i)$ and corresponding exchanges in $M'_2(n,i)$ and go to Step 4.

- *Step 3* Set $i = i + 1$. If $i \pmod{I}$ is equal to 0, go to Step 4; if $i \pmod{I}$ is greater than 0, go back to Step 1.

- *Step 4* Set $n = n - 1$. If $n > 0$, go back to Step 1; if $n = 0$, stop.

With this algorithm, one can get the markets for sale in $M_1'(n, i)$, and the corresponding selling thresholds-targets in $U(n, i)$, when the period state at period n is i,. One can also get the markets for purchase in $M_2'(n, i)$, and the corresponding selling thresholds-targets in $B(n, i)$.

□ **End of chapter.**

Chapter 5

Conclusion

In this thesis we develop an inventory trading model for the companies that consume commodities such as natural gas, crude oil and their products. We consider a discrete-time, single-item, single-location inventory system which can trade in different spot exchanges. And the prices of these exchanges are influenced by an exogenous Markov process. The unique aspect of our model is the integration of a multi-source model and price dynamic model. We characterize the structure of our model's optimal inventory trading policy, which is determined by minimizing the expected cost, and find the conditions to ensure that the generalized (s, S, H) policy is optimal. We also give an algorithm to compute the thresholds and targets of this optimal policy.

Our work can lend itself to the storable-commodity industries, which are characterized by spot-price fluctuation. For instance, one can use our model to optimize gas loading strategy for the gas fired generator.

One can also apply the optimal inventory trading policy at the downstream storage facilities in the energy value chain, and assess these storage facilities by quantifying the benefits generated from them. Further more, our model could be used to evaluate the leasing contracts on the downstream storage facilities.

Finally, the value chain of energy commodities such as gas and oil always constitutes a seller and a buyer whose commercial relationships are governed by agreements and contracts. In our work we consider the benefit of the companies that consume the commodities based on exogenously specified price Markov process. Yet the study of contractual issues related to the interactions between sellers and buyers is a broader area for future research.

□ **End of chapter.**

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